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On the Iteration of a Rational Function: Computer Experiments with Newton's Method

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Abstract. Using Newton's method to look for roots of a polynomial in the complex plane amounts to iterating a certain rational function. This article describes the behavior of Newton iteration for cubic polynomials. After a change of variables, these polynomials can be parametrized by a single complex parameter, and the Newton transformation has a single critical point other than its fixed points at the roots of the polynomial. We describe the behavior of the orbit of the free critical point as the parameter is varied. The Julia set, points where Newton's method fail to converge, is also pictured. These sets exhibit an unexpected stability of their gross structure while the changes in small scale structure are intricate and subtle.

1. Introduction

The study of iteration of rational mappings of a complex variable has a long history. Seminal work on this topic appeared in the early studies of Fatou and Julia at the turn of the 20th century [4, 7]. In the 60's the work of Brolin [1] and Guckenheimer [5], Jakobson [6] should be mentioned. More recently Sullivan [10] Mané et al. [9] have made contributions. The articles cited above have made significant theoretical contributions; in contrast there have been few experimental studies of the iterates of rational maps. The work of Mandelbrot [8] has stood alone.

Mandelbrot has considered iteration of the transformation

$$f_c(z) = z^2 + c, \tag{1.1}$$

where c is a complex number, and produced striking pictures whose most obvious feature is the prevailing self-similarity. A major contribution toward the understanding of Mandelbrot's bifurcation diagram has been announced by Douady and Hubbard [2, 3].