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Degenerate Asymptotic Perturbation Theory

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Abstract. Asymptotic Rayleigh–Schrödinger perturbation theory for discrete eigenvalues is developed systematically in the general degenerate case. For this purpose we study the spectral properties of $m \times m$—matrix functions $A(\kappa)$ of a complex variable κ which have an asymptotic expansion $\Sigma A_k \kappa^k$ as $\kappa \to 0$. We show that asymptotic expansions for groups of eigenvalues and for the corresponding spectral projections of $A(\kappa)$ can be obtained from the set $\{A_k\}$ by analytic perturbation theory. Special attention is given to the case where $A(\kappa)$ is Borel-summable in some sector originating from $\kappa = 0$ with opening angle $> \pi$. Here we prove that the asymptotic series describe individual eigenvalues and eigenprojections of $A(\kappa)$ which are shown to be holomorphic in S near $\kappa = 0$ and Borel summable if $A_k^* = A_k$ for all k. We then fit these results into the scheme of Rayleigh–Schrödinger perturbation theory and we give some examples of asymptotic estimates for Schrödinger operators.

Introduction

Convergent (or analytic) perturbation theory has long been established in full generality by Kato [9] and Baumgärtel [3]. Yet, among the simplest problems of quantum mechanics, there are cases where the perturbation series diverge and have only an asymptotic validity: the anharmonic oscillator [10], the Stark–effect [7] and the Zeeman–effect [2]. In all these examples the perturbed eigenvalues can be constructed as the Borel–sum of their divergent asymptotic series.

These beautiful results are not quite satisfactory in one respect: they suffer from the unnatural assumption that the unperturbed eigenvalue is non-degenerate (or, in some cases, that its degeneracy is lifted in first order). In their accounts of asymptotic perturbation theory Reed and Simon [10] tacitly ignore the degenerate case while Kato ([9], Chap. 8) only discusses low order perturbations of semi-simple eigenvalues. Our aim is to remove restrictions of this kind and to develop asymptotic perturbation theory in the general degenerate case to any possible order.

Not surprisingly it turns out that the heart of the matter is the finite-dimensional case. Given an $m \times m$—matrix valued function $A(\kappa)$ of a complex variable κ which