Commun. Math. Phys. 88, 479–501 (1983)

Communications in Mathematical Physics

On the Invariant Sets of a Family of Quadratic Maps

M. F. Barnsley*, J. S. Geronimo**, and A. N. Harrington

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

Abstract. The Julia set B_λ for the mapping $z \to (z-\lambda)^2$ is considered, where λ is a complex parameter. For $\lambda \geqq 2$ a new upper bound for the Hausdorff dimension is given, and the monic polynomials orthogonal with respect to the equilibrium measure on B_λ are introduced. A method for calculating all of the polynomials is provided, and certain identities which obtain among coefficients of the three-term recurrence relations are given. A unifying theme is the relationship between B_λ and λ-chains $\lambda \pm \sqrt{(\lambda \pm \sqrt{(\lambda \pm \ldots)}}$, which is explored for $-\frac{1}{4} \leqq \lambda \leqq 2$ and for $\lambda \in \mathbb{C}$ with $|\lambda| \leqq \frac{1}{4}$, with the aid of the Böttcher equation. Then B_λ is shown to be a Hölder continuous curve for $|\lambda| < \frac{1}{4}$.

1. Introduction

In this paper we consider the Julia set B_λ for the mapping

$$T_\lambda z = (z-\lambda)^2, \qquad z \in \mathbb{C},$$

of the complex plane into itself, where λ is a parameter which may be real or complex. Here T_λ is equivalent to $z \to 1 - \lambda z^2$ which has been studied in the context of iterated maps of intervals, see [10, 13], and also to $z \to z^2 + \lambda$, see [11].

B_λ was first studied by Fatou [12] and Julia [19] in the context of arbitrary rational transformations. With the notation

$$T_\lambda^0 z = z, \quad \text{and} \quad T_\lambda^{n+1} z = T_\lambda(T_\lambda^n z) \quad \text{for} \quad n \in \{1, 2, 3, \ldots\},$$

B_λ can be defined to be those points in $\mathbb{C}$ where $\{T_\lambda^n z\}$ is not normal. This is the starting point of the survey by Brolin [8]. Equivalently B_λ can be defined to be the closure of the set of all repulsive k-cycles, $k \in \{1, 2, 3, \ldots\}$, [12]. This shows at once the relevance of B_λ to the corresponding iterated real map where $B_\lambda \cap \mathbb{R}$ plays a central role.

* Supported by NSF Grant MCS-8104862

** Supported by NSF Grant MCS-8002731