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On Zamolodchikov's Solution of the Tetrahedron Equations

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Abstract. The tetrahedron equations arise in field theory as the condition for the S-matrix in $2+1$-dimensions to be factorizable, and in statistical mechanics as the condition that the transfer matrices of three-dimensional models commute. Zamolodchikov has proposed what appear (from numerical evidence and special cases) to be non-trivial particular solutions of these equations, but has not fully verified them. Here it is proved that they are indeed solutions.

1. Introduction

A number of two-dimensional models in statistical mechanics have been exactly solved [1–4] by using the "star-triangle equations" (or simply "triangle equations") [5–7], which are generalizations of the star-triangle relation of the Ising model [8, 9]. These equations are the conditions for two row-to-row transfer matrices to commute.

Alternatively, these models can be put into field-theoretic form by considering the transfer matrix that adds a single face to the lattice [10], and regarding this as an S-matrix. The star-triangle relations then become the condition for the S-matrix to factorize [11].

These equations can be generalized to three-dimensional models in statistical mechanics, corresponding to a $1+2$-dimensional field theory. Unfortunately, the resulting "tetrahedron" equations are immensely more complicated, the main problem being that there are 2^{14} individual equations to satisfy for an Ising-type model, as against 2^6 in two-dimensions. Symmetries reduce this number somewhat, but there are still apparently many more equations than unknowns and until recently there was little reason to suppose that the equations permitted any interesting solutions at all.

However, by what appears to be an extraordinary feat of intuition, Zamolodchikov [12, 13] has written down particular possible solutions and has shown that they satisfy some of the tetrahedron equations in various limiting cases. Extensive numerical tests have also been made by V. Bajanov and Yu.