Commun. Math. Phys. 86, 529–538 (1982)

Communications in
Mathematical
Physics

Measures on Projections and Physical States

Erik Christensen

Matematisk Institut, Københavns Universitet, DK-2100 København, Danmark

Abstract. It is shown that a finitely additive measure on the projections of a von Neumann algebra without I_2 and II_1 summands is the restriction of a state. A definition of a physical state is proposed, and it is shown that such a physical state on a simple C^*-algebra with unit is a state.

1. Introduction

In the classical mathematical model for quantum mechanics one has the notions of "yes-no experiment" and "state". The state of the system is a certain preparation of it, and one associates to each experiment a value in each state, namely the probability for yes in this state, and this meanvalue is then called the value of the state on the proposition. As a more detailed description of the set of propositions including their lattice properties, one uses operator algebras, and one supposes that the propositions correspond to projections in a noncommutative self-adjoint algebra of operators on a Hilbert-space [5, 6]. The value of the state on the union of two compatible disjoint propositions is the sum of the values on each, and since disjoint compatible propositions correspond to pairwise orthogonal projections, we find that the state is transformed into a function on the set of projections in the algebra, which is additive on any finite set of pairwise orthogonal projections. Such a function is from now on called a measure on projections.

In [6] Mackey asked whether a countably additive measure is the restriction of a positive linear functional on the algebra – just as in ordinary integration theory – and shortly after, Gleason [3] settled the question in the affirmative for the algebra of all bounded operators on a Hilbert space of dimension at least three. Various people have worked on the problem and we want to mention the works by Aarnes [1] and Gunson [4], upon which this paper is built. Gunson proves that countably additive measures on hyperfinite II_1 factors are restrictions of states, by using showing that such measures have certain continuity properties, and then Gleason's result on the dense algebra consisting of a union of an increasing sequence of finite dimensional subfactors. The fundamental lemma Gunson uses is stated in Lemma 2.3 and it is also crucial here. Aarnes does not study the measure

0010-3616/82/0086/0529/$02.00