Commun. Math. Phys. 86, 515–527 (1982)

Communications in Mathematical Physics

A Direct Method for Minimizing the Yang-Mills Functional over 4-Manifolds

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Abstract. A direct method is employed to minimize the Yang-Mills functional over a 4-dimensional manifold. The limiting connection is shown to be Yang-Mills, but in a possibly new bundle. We show that a topological invariant of the bundle is preserved by the minimizing process. This implies the existence of an absolute minimum of the Yang-Mills functional in a wide class of bundles.

Introduction

We examine the limiting behavior of a minimizing sequence of connections for the Yang-Mills functional in a principal bundle over a compact 4-manifold. A limiting connection is found, but possibly in a new bundle. It is natural to ask for some invariant of the bundle which is preserved by this procedure. In the minimizing process, there are a finite number of points where curvature collects. When we take the limit, we lose control of the bundle at these points. So an invariant which will survive through the limit should be determined by the bundle with a finite number of fibers removed. If the invariant is to be in cohomology, we see that we want classes which are determined by their restriction to the manifold with finitely many points removed. For 4 manifolds, this is satisfied by 2- and 3-dimensional cohomology classes. Uhlenbeck makes a conjecture in [17] that the first chern class of a unitary bundle is preserved under the minimizing process. Our results show that the conjecture is true, although this case does not seem the most important application of our results.

In his paper [13] Taubes shows existence of self-dual Yang-Mills fields on many oriented 4-manifolds. The principle bundles to which his method applies must have an invariant in dimension 2 cohomology vanishing. This invariant is the obstruction to lifting the structure group of a principle bundle to the universal covering group of the structure group. In this paper we show that this obstruction is preserved by our process, so we obtain Yang-Mills fields in bundles with nontrivial obstruction.

This obstruction also arises in 't Hooft's [14] construction of bundles over a 4-dimensional torus with structure group $SU(n)$ modulo its center. By explicit

0010-3616/82/0086/0515/$02.60