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Non-Uniqueness of Solutions of Percival's Euler-Lagrange Equation

John N. Mather*

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Abstract. Percival [5, 6] introduced a Langrangian and an Euler-Lagrange equation for finding quasi-periodic orbits. In [3], we studied area preserving twist homeomorphisms of the annulus, using Percival's formalism. We showed that Percival's Lagrangian has a maximum on a suitable function space, and that a point where it takes its maximum is a solution of Percival's Euler-Lagrange equation. Moreover, in the rigorous interpretation of Percival's formalism which we gave in [3], the solutions of Percival's Euler-Lagrange equation correspond bijectively to a certain class of minimal sets. (We will prove this in Sect. 2.) In [4], we showed that Percival's Lagrangian takes its maximum at only one point. In this paper, we show that there exist C^∞ area preserving twist diffeomorphisms of the annulus, for which there exists at least one solution of Percival's Euler-Lagrange equation where Percival's Lagrangian does not take its maximum. In other words, solutions of Percival's Euler-Lagrange equation need not be unique.

1. Statement of the Results

In this section, we recall basic notations, terminology, and results from [3] and [4], and state the theorem which we will prove in this paper.

We will denote by $\mathscr{F}$ the class of homeomorphisms which was considered in [3]. This is defined as follows.

We set $A=\{(x,y)\in\mathbb{R}^2 : 0\leqq y\leqq 1\}$. We let $T: A\to A$ be the translation $T(x,y)=(x+1,y)$. We let $\mathscr{F}$ be the set of homeomorphisms of A which satisfy the following two conditions:

1) f is area preserving, orientation preserving, boundary component preserving, and $fT=Tf$.

2) (positive monotone twist condition) $f(x,y)_1>f(x,z)_1$ if $y>z$. Here, p_1 denotes the first coordinate of p, if $p\in A=\mathbb{R}\times[0,1]$.

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