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# The Existence of a Non-Minimal Solution to the SU (2) Yang-Mills-Higgs Equations on $\mathbb{R}^3$. Part I

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**Abstract.** This paper (Part I) and the sequel (Part II) prove the existence of a smooth, non-trivial, finite action solution to the SU (2) Yang-Mills-Higgs equations on $\mathbb{R}^3$ in the Bogomol'nyi-Prasad-Sommerfield limit. The proof uses a simple form of Morse theory known as Ljusternik-Šnirelman theory. Part I establishes that a form of Lusternik-Šnirelman theory is applicable to the SU (2) Yang-Mills-Higgs equations. Here, a sufficient condition for the existence of the aforementioned solution is derived. Part II contains the completed existence proof. There it is demonstrated that the sufficient condition of Part I is satisfied by the SU (2) Yang-Mills-Higgs equations.

## I. Introduction

The SU (2) Yang-Mills-Higgs equations on $\mathbb{R}^3$ are the variational equations for a connection (the Yang-Mills potential) and a minimally coupled, associated scalar field which transforms according to the adjoint representation of SU (2) (the Higgs field). These are the variational equations of an action functional [see Eq. (2.1)]. The equations become interesting when one requires the action to be finite, and the boundary condition that the Higgs field have unit norm, asymptotically on $\mathbb{R}^3$, see Eqs. (2.2) and (2.3). This is the Bogomol'nyi-Prasad-Sommerfield limit. In addition there is a first-order system of equations which characterize minima of the functional (2.1); these are called the Bogomol'nyi equations (2.6). As minima, every solution to (2.6) also satisfies the second-order equations (2.2) and (2.3). This general set-up has an analogy with Yang-Mills theory on $S^4$ [1] and also with Ginzburg-Landau theory [2] ($\lambda = 1$) on $\mathbb{R}^2$. Both these have second-order variational equations and associated first-order equations for minima. The following conjecture has been made for the Yang-Mills-Higgs equations (2.2) and (2.3), the $\lambda = 1$ Ginzburg-Landau equations on $\mathbb{R}^2$ and the Yang-Mills equations on $S^4$: Every finite action solution to the variational equations is a minima; hence

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