Commun. Math. Phys. 86, 143–147 (1982)

Communications in
Mathematical
Physics

Brownian Motion in a Convex Ring and Quasi-Concavity

Christer Borell
Chalmers University of Technology, S-41296 Göteborg, Sweden

Abstract. Let X be the Brownian motion in $\mathbb{R}^n$ and denote by τ_M the first hitting time of $M \subseteqq \mathbb{R}^n$. Given convex sets $K \subseteqq L \subseteqq \mathbb{R}^n$ we prove that all the level sets

$$\{(x,t) \in \mathbb{R}^n \times [0,+\infty[\,; P_x[\tau_K \leqq t \wedge \tau_{L^c}] \geqq \lambda\},\ \lambda \in \mathbb{R},$$

are convex.

1. Introduction

The background of the present paper is a very beautiful theorem of Gabriel [3, 4] and Lewis [5] stating that the equilibrium potential of a convex body in $\mathbb{R}^n$ relative to a surrounding convex body is quasi-concave. Below we will show the same property for the solution of the corresponding heat conduction problem with zero initial data. Here recall that a real-valued function f defined on a convex set is said to be quasi-concave if all the level sets $\{f \geqq \lambda\}$, $\lambda \in \mathbb{R}$, are convex.

Throughout, X denotes the Brownian motion in $\mathbb{R}^n$ and, for each $M \subseteqq \mathbb{R}^n$, τ_M stands for the first hitting time of M, that is, $\tau_M = \inf\{t > 0; X(t) \in M\}$.

Theorem 1.1. *Suppose $K, L \subseteqq \mathbb{R}^n$ are convex sets such that $K \subseteqq L$. Then the function*

$$w(x,t) = P_x[\tau_K \leqq t \wedge \tau_{L^c}], \qquad (x,t) \in \mathbb{R}^n \times [0,+\infty[\,,$$

is quasi-concave.

Here, for short, L^c means $\mathbb{R}^n \backslash L$.

To prove Theorem 1.1 there is no loss of generality to assume that (i) K is a convex body in $\mathbb{R}^n$, (ii) L is the interior of a convex body in $\mathbb{R}^n$, and (iii) $d(K, L^c) > 0$. In what follows, we always assume (i)–(iii) are fulfilled. Then, in particular,

$$\begin{cases} \Delta w = 2w'_t & \text{in} \quad (L\backslash K) \times]0,+\infty[\\ w = 0 & \text{on} \quad \{(L\backslash K) \times \{0\}\} \cup \{\partial L \times [0,+\infty[\} \\ w = 1 & \text{on} \quad \partial K \times [0,+\infty[\end{cases}$$

0010-3616/82/0086/0143/$01.00