

The Virasoro Algebra

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Abstract. Three theorems are proved. With suitable hypotheses in each case, characterizations are found for the Virasoro algebra, for some of its representations, and for the Ramond–Neveu–Schwarz superalgebra built around the Virasoro algebra.

1. Introduction

In its centerless version the Virasoro algebra is a Lie algebra with basis u_i , i ranging over the integers, and multiplication table $u_i u_j = (i - j)u_{i+j}$. (I am omitting brackets throughout the paper in the belief that there will be no ambiguity.) It surfaced in the physics literature in the late 1960's and numerous later physics papers have studied it and related algebras.

There was (more or less) an anticipation in the mathematical literature. In the 1930's Witt discovered the Lie algebra that subsequently carried his name; it has the same multiplication table as the Virasoro algebra, but the subscripts range over the integers mod p , p a prime, and the characteristic of the coefficient field is p . Witt did not publish anything; the first reference occurred in the paper [6] of Zassenhaus. In the early 1950's several mathematicians discussed the centerless Virasoro algebra (calling it the "infinite Witt algebra") but nothing was published.

Ramond [4] discovered a way of draping a Lie superalgebra around the Virasoro algebra. Independently, and about at the same time, Neveu and Schwarz [3] did it in a slightly different way. These papers marked the first appearance of Lie superalgebras in the western physics literature.

In this paper I make several modest contributions. I give a certain characterization of the Virasoro algebra; I study some of its representations; and I establish a uniqueness theorem for the Ramond–Neveu–Schwarz construction.

In [5] there is a uniqueness theorem for superalgebras which are allowed to be considerably larger. It is to be observed that they assume outright that the products follow the familiar pattern. The point of this paper is to show, under suitable hypotheses, that this pattern is forced in the "small" case.