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Stability Theory for Solitary-Wave Solutions of Scalar Field Equations

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Abstract. We prove stability and instability theorems for solitary-wave solutions of classical scalar field equations.

0. Introduction

In this paper, we study the stability of special travelling-wave ("solitary-wave", [1]) solutions of classical scalar field equations of the form

$$\Box\phi + U'(\phi) = 0. \tag{0-1}$$

This problem has attracted much attention recently in the physics literature ([2]), in part because classical solutions may be recovered from suitable expectation values of quantum fields in the classical limit ([3]).

Apart from the main motivation, which is to provide a simple and clear mathematical theory of stability for classical field equations, there is also a three-fold physical motivation. Firstly, most of the discussion in the physics literature ([2]), which is heuristically correct, relies on the linear theory. It may be shown, however, using methods of the present paper, that the latter is not applicable, because the linearized operator (on the natural Hilbert space, after proper "subtraction" of the zero mode) is skew-adjoint, a reflection of the fact that the mechanism of stability in these theories is dispersive, not dissipative (see also the discussion in [6] for K-dV equation). Secondly, the existing rigorous nonlinear stability theories ([4], generalized and corrected in [5], and [6]) are in principle applicable only to a class of equations (such as the K-dV equation) which may be treated either by inverse scattering theory ([6]), or which possess more than one scalar conservation law ([4], [5]), and are, therefore, unsuitable to describe, for instance, the stability of "kinks" of the nonlinear Klein-Gordon equation ([2]). Thirdly, and perhaps most importantly, the heuristic discussion disregards the somewhat delicate technical problems posed

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