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The Rotation Number for Almost Periodic Potentials

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Abstract. We define and analyze the rotation number for the almost periodic Schrödinger operator $L = \frac{-d^2}{dx^2} + q(x)$. We use the rotation number to discuss (i) the spectrum of L; (ii) its relation to the Korteweg-de Vries equation.

1. Introduction

Almost Periodic Potentials

The spectral theory for second order differential operators

$$L\varphi = \left(-\frac{d^2}{dx^2} + q(x) \right) \varphi = \lambda\varphi \tag{1.1}$$

on the x-axis $(-\infty, \infty)$ is well understood, having been developed by H. Weyl in 1910. In particular, if $q(x)$ is bounded (which is the case when $q(x)$ is almost periodic), then one has the limitpoint case at ∞ and at $-\infty$. The nature of the spectrum $\sigma(L)$, however, is not as well understood; it depends rather subtly on the asymptotic behavior of $q(x)$ for large $|x|$. For periodic potentials $q(x)$-in this case one speaks of the "Hill's equation" – it is well known that the spectrum is continuous and consists of finitely or infinitely many intervals, the so-called band spectrum. These facts can be deduced from the Floquet theory, which describes the behavior of the solutions of any system with periodic coefficients.

We are interested in the case of almost periodic potentials in the sense of H. Bohr. In this case there is no such elementary Floquet theory, and little is known about the nature of the spectrum. It certainly can have features which do not occur in the periodic case; for example, one can have point eigenvalues, and one can have nowhere dense spectrum (see [17]).

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