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Spectral Behavior of Quasi Periodic Potentials*

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Abstract. Spectral properties of Schrödinger operators of the type $H_\varepsilon = -\Delta + \varepsilon V$, where Δ is the Laplacian, V a quasiperiodic potential and ε a coupling constant, are developed. V is taken to be finite sum of exponentials with generic frequencies. For small ε a strong stability is shown. On the other hand, examples (in the finite difference case) are given, for which a transition in the type of spectrum occurs, as ε is increased.

0. Introduction

Not much is known about the spectral analysis of operators of the type $H = -\Delta + V$ (where Δ is the Laplacian and V is some almost periodic potential) acting on $L^2(\mathbf{R}^\nu)$ or $l^2(\mathbf{Z}^\nu)$. The spectral analysis of such operators in the case of $\mathbf{R}^1$ could be relevant in understanding the rings of Saturn (Avron-Simon [2]) and is also of interest in insulator-conductor models of materials, Aubry [1]. The case of a random potential is of considerable interest in physics (see Thouless [10]). In any case the spectral behavior of the continuous and discrete cases are believed to be similar, and we will discuss various aspects of both of them.

In spectral analysis there are two basic questions which are of interest.

(1) What is the spectrum as a set in the plane? (We denote this set by $\sigma(H)$.)

(2) What is the nature of the eigenstates? More precisely are the eigenstates localized (or pure point spectrum) or extended (continuous spectrum) or some mixture? This latter question is the one which determines the insulator-conductor properties of the material.

We will consider more generally

$$H_\varepsilon = -\Delta + \varepsilon V, \tag{0.1}$$

and examine how the spectrum depends on ε. The coupling constant ε would correspond to the reciprocal of the temperature, in the physical models. One of the results of this paper is to give some examples of almost periodic V for which the spectrum exhibits a phase transition from extended to localized states.

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