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Analytic Structure and Explicit Solution of an Important Implicit Equation

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Abstract. The equation $z = 2G(z) - \exp G(z) + 1$ (and similar ones obtained from it by substitutions) appears in connection with a variety of problems ranging from pure mathematics (combinatorics; some first order, nonlinear differential equations) over statistical thermodynamics to renormalization theory. It is therefore of interest to solve this equation for $G(z)$ explicitly. It turns out, after study of the complex structure of the z and G planes, that an explicit integral representation of $G(z)$ can be given, which may be directly used for numerical calculations of high precision.

1. Introduction

The equation to be studied in this paper [called the "bootstrap equation" (BE)], namely

$$z = 2G(z) - \exp G(z) + 1 \tag{1.1}$$

can, by substitutions, be brought into various forms. Take, for instance, the substitution

$$\begin{aligned} z &= f(w), \\ G(z) &= A + B \cdot H(w), \end{aligned} \tag{1.2}$$

which yields

$$\begin{aligned} f(w) &= 2B \cdot H(w) - C\exp[B \cdot H(w)] + D, \\ C &:= e^A; \; D := 2A + 1 . \end{aligned} \tag{1.3}$$

Substituting further $H(w) = \ln J(\mathrm{w})$ or any other function which can be explicitly inverted, one arrives at a large variety of equations which are equivalent to Eq. (1.1). We discuss therefore, without loss of generality, the solution of Eq. (1.1) as a representative of a whole class of equations.

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