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Horseshoes in Perturbations of Hamiltonian Systems with Two Degrees of Freedom*

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Abstract. This paper concerns Hamiltonian and non-Hamiltonian perturbations of integrable two degree of freedom Hamiltonian systems which contain homoclinic and periodic orbits. Our main example concerns perturbations of the uncoupled system consisting of the simple pendulum and the harmonic oscillator. We show that small coupling perturbations with, possibly, the addition of positive and negative damping breaks the integrability by introducing horseshoes into the dynamics.

1. Introduction

This paper concerns Hamiltonian and non-Hamiltonian perturbations of integrable two degree of freedom Hamiltonian systems which contain homoclinic and periodic orbits. Our main example concerns perturbations of the uncoupled system consisting of the simple pendulum and the harmonic oscillator. We show that small coupling perturbations with, possibly, the addition of positive and negative damping breaks the integrability by introducing horseshoes into the dynamics.

We begin with an unperturbed $n+1$ degree of freedom Hamiltonian in canonical coordinates $q=(q^1, \ldots, q^n)$, $p=(p_1, \ldots, p_n)$, x, y of the form

$$H^0(q,p,x,y)=F(q,p)+G(x,y). \tag{1.1}$$

Starting in Sect. 3, we will assume $n=1$, but for some of the development n can be arbitrary. Allowing x and y to be multidimensional will be the subject of another publication.

We shall assume that G admits action-angle variables; i.e. there is a canonical change of coordinates to (θ, I) such that θ is 2π periodic, $I \geqq 0$ and G becomes a function of I alone; we write $G(I)$ for this function and assume that

$$G(0)=0, \qquad \Omega(I) \equiv G'(I) > 0 \quad \text{for} \quad I > 0. \tag{1.2}$$

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