Commun. Math. Phys. 82, 497–509 (1982)

Communications in Mathematical Physics

Pure States as a Dual Object for C^*-Algebras

Frederic W. Shultz*
Wellesley College, Wellesley, MA 02181, USA

Abstract. We consider the set of pure states of a C^*-algebra as a uniform space equipped with transition probabilities and orientation, and show that the pure states with this structure determine the C^*-algebra up to *-isomorphism.

Introduction

For commutative unital C^*-algebras, it is well known that the set of pure states (as a topological space) determines the algebra. In fact, any two such algebras are isomorphic to $C(X)$ and $C(Y)$ for compact Hausdorff spaces X and Y. The pure states of $C(X)$ are just evaluation at each x in X, and every homeomorphism of Y onto X is induced by a *-isomorphism of $C(X)$ onto $C(Y)$. The Stone–Weierstrass theorem is a special case of this.

For general C^*-algebras it is clear that this result fails, e.g. not every homeomorphism of the pure states $P(B)$ onto $P(A)$ is induced by a *-isomorphism; $P(A)$ (as a topological space) does not determine A. The purpose of this paper is to show that $P(A)$ does determine A if given a suitable structure.

The roots of our investigation go back to the work of Kadison [12–14] and Wigner [19]. Kadison studied the representation of a C^*-algebra as continuous functions on $P(A)$ (or $P(A)^-$). He showed [13] that a homeomorphism of $P(B)^-$ onto $P(A)^-$ which carries A onto B is induced by a Jordan isomorphism. Wigner focused on transition probabilities between pure states. He showed that a bijection of the pure normal states of $B(H_2)$ onto those of $B(H_1)$ which preserves transition probabilities is induced by a Jordan isomorphism (in this case, a *-isomorphism or *-anti-isomorphism) of $B(H_1)$ onto $B(H_2)$. There have also been investigations of Stone–Weierstrass theorems for C^*-algebras, e.g. Kaplansky [16], Glimm [11], Sakai [17]; Akemann [1, 2], Giles and Kummer [9], and Effros [8].

The recent work from which this paper springs is joint work with Alfsen and Hanche-Olsen [5], in which the notion of orientation of a state space was introduced. It was shown there that an affine homeomorphism of state spaces is induced by a *-isomorphism iff the map preserves orientation.

Our work combines the structures of topology (or uniformity), transition

* Partially supported by NSF grant MCS78-02455

0010-3616/82/0082/0497/$02.60