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Integrable Nonlinear Equations and Liouville's Theorem, II

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Abstract. A symplectic structure is constructed and the Liouville integration carried out for a stationary Lax equation $[L, P] = 0$, where L is a scalar differential operator of an arbitrary order. n^{th} order operators are included into the variety of first-order matrix operators, and properties of this inclusion are studied.

This article is in fact the third part of the work ([1, 2]) although the last two parts are independent of the first. Here we deal with equations arising from an n^{th} order linear differential operator. For simplicity we consider the scalar case only, this restriction is not of great importance. The integration of such equations was carried out first by Kritchever ([3, 4]) and his work relates to this article as our previous paper relates to the work of Dubrovin ([5, 6]). But this time the connection is much weaker. Our method does not resemble Kritchever's, in particular since we use different variables, it is even difficult to compare the results. We use the reduction of an n^{th} order differential operator to a first order matrix operator (in an Appendix we discuss this reduction in more details than are needed for this article). After this reduction, the further development is close to that of the previous article, however with essential differences. We pay more attention to these differences, as often as possible replacing detailed proofs by references to [2].

1. We start off with the equation

$$-Q' + [U + \zeta A, Q] = 0 \tag{1}$$

where Q, U, A are $n \times n$ matrices, $\zeta = z^n$ a complex parameter, A and U have the form

$$A = \begin{pmatrix} 0 & & \\ \vdots & \ddots & \\ 1 & \dots & 0 \end{pmatrix} \quad U = \begin{pmatrix} 0 & 1 & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & 1 \\ -u_0, & \dots, & & -u_{n-2}, & 0 \end{pmatrix}.$$

$u_0, \dots, u_{n-2}$ will be taken as independent generators of a differential algebra $\mathscr{A}$ (which consists of polynomials in $u_i^{(k)}$ with complex coefficients). The matrix Q is a solution we are looking for. We shall numerate rows and columns from 0 to $n-1$.

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