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Almost Periodic Schrödinger Operators

I. Limit Periodic Potentials*

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Abstract. We study $H = -d^2/dx^2 + V(x)$ with $V(x)$ limit periodic, e.g. $V(x) = \sum a_n \cos(x/2^n)$ with $\sum |a_n| < \infty$. We prove that for a generic V (and for generic a_n in the explicit example), $\sigma(H)$ is a Cantor (≡nowhere dense, perfect) set. For a dense set, the spectrum is both Cantor and purely absolutely continuous and therefore purely recurrent absolutely continuous.

1. Introduction

This is the first of several papers on the spectral properties of operators $-d^2/dx^2 + V(x)$ (and its higher dimensional analogs) with $V(x)$ an almost periodic function. Two themes will recur throughout:

(1) There is a tendency for the spectrum to be a Cantor set (≡nowhere dense, closed set with no isolated points), albeit one with positive Lebesgue measure.

(2) If V is multiplied by a suitably large constant, there are "mobility edges", in the sense that the spectrum in certain intervals is pure point and otherwise is absolutely continuous (however, if (1) holds the absolutely continuous spectrum must be recurrent in the sense of [2] so that the states are not exactly "mobile"; since Cantor sets are locally uncountable, the point spectrum will be "thick" in the sense of [2]).

We emphasize that while we believe both these phenomena occur for certain almost periodic potentials, we have not yet proven this. In the present paper, we prove (1) for generic limit periodic potentials. We recall

Definition. A function $V(x)$ on $(-\infty, \infty)$ is called *limit periodic* if there exist continuous periodic functions $V_n(x)$ of period L_n so that $\sup_x |V_n(x) - V(x)| \to 0$ as $n \to \infty$.

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