Commun. Math. Phys. 81, 327–340 (1981)

Communications in
Mathematical
Physics

A Mass Zero Cluster Expansion

Part 1. The Expansion*

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Abstract. A cluster expansion is developed and applied to study the perturbation $\lambda(\Delta\phi)^4$ of the massless lattice field ϕ in dimension 3. The method is loosely inspired by the work of Gawedzki and Kupiainen on block spin techniques for the $\lambda(\vec{\nabla}\phi)^4$ system. The cluster expansion is given in terms of expansion coefficients for the field as a sum of certain special block spin functions. These functions are chosen with a large number of moments zero, so that the interaction couples spatially separated functions with an interaction falling off as a high inverse power of the separation distance. The present techniques, with some technical development, should work for broad classes of other models, including the lattice dipole gas and the $\lambda(\vec{\nabla}\phi)^4$ model. Models $\lambda(|\Delta|^\alpha\phi)^{2s}, \alpha > \frac{1}{2}$, are essentially included in the present work.

1. Introduction

Cluster expansions have had a wide variety of applications in field theory and statistical mechanics. They have been used to prove the existence of correlation functions in the thermodynamic limit, properties of the spectrum, clustering of correlation functions, the existence of phase transitions, Debye screening, and many other detailed properties of very diverse systems. Typically they may be applied to systems where such special techniques as correlation inequalities, reflection positivity, and Markov properties may not apply. It is of obvious interest to extend the applicability of cluster techniques to systems with zero masses—systems with long range interactions.

In this paper we develop a new cluster expansion and apply it to the lattice model with Hamiltonian density $\frac{1}{2}(\nabla\phi)^2 + \lambda(\Delta\phi)^4$ in 3 dimensions. Other models, such as the lattice dipole gas (or continuum dipole gas) or the $\lambda(\vec{\nabla}\phi)^4$ model, should be accessible to similar treatment with increased technical difficulties. We intend to come back to study the dipole gas in a later paper. We do not know if the present techniques will be useful in studying models such as the rotator model or the Heisenberg model. Part 1 of this paper introduces the general concepts, defines the cluster expansion, and contains basic estimates, all of which should be useful in further applications. Part 2 contains the combinatoric details involved in proving the convergence of the cluster expansion.

* This work was supported in part by the National Science Foundation under Grant No. PHY 79-05688

0010-3616/81/0081/0327/$02.80