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Stochastic Operators, Information, and Entropy

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Abstract. For a stochastic operator U on an L_1-space, i.e. U is linear, positive, and norm preserving on the positive cone of L_1, it is shown that U decreases relative information between two nonnegative L_1-functions. Furthermore it is shown that the following properties of U are closely related: U is energy decreasing (energy preserving), U is H-decreasing, where H is Boltzmann's H-functional, and the Maxwell distributions are fixed points of U.

The aim of this note is to prove some properties of stochastic operators on L_1-spaces. In Sect. 1 we show that a stochastic operator decreases relative information between two nonnegative L_1-functions. Such a property was known for special cases.

In Sect. 2 we show that, for a stochastic operator U, certain properties are equivalent. If α is a function on the measure space defining the energy and H is Boltzmann's H-functional, then, for instance, it is shown that U is energy decreasing and H-decreasing if and only if all "Maxwell distributions" $\exp(-\kappa\alpha)$ $(\kappa \geqq 1)$ are invariant under U. These properties are also equivalent to the property that U is energy preserving and leaves one "Maxwell distribution" $\exp(-\alpha)$ fixed.

In [13], the author proves the H-theorem for Boltzmann type equations $u' = Tu + J(u)$ in $L_1(\mu)$, for some measure space $(\Omega, \mathscr{A}, \mu)$. The required conditions are posed in abstract form on the strongly continuous semigroup $(U(t);\ t \geqq 0)$ of "free motion" generated by T, and on the "collision operator" J separately. In applications, $U(t)$ should be expected to be a stochastic operator for each $t \geqq 0$. As a consequence of Theorem 2.1 and Proposition 2.5, one can obtain relations between some of the conditions for $(U(t))$; this is discussed in [13, remarks preceding Proposition 3.1]. As an example we consider $\Omega = D \times \mathbb{R}^3$, where $D \subset \mathbb{R}^3$ is open (and has suitable boundary), μ is Lebesgue measure, and T is an operator associated with the differential expression $-\xi \cdot \mathrm{grad}_x$ and a

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