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On the Existence of Invariant, Absolutely Continuous Measures

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Abstract. Let $(\Omega, \mathfrak{B}, \lambda)$ be a measure space with normalized measure, $f: \Omega \to \Omega$ a nonsingular transformation. We prove: there exists an f-invariant normalized measure which is absolutely continuous with respect to λ *if and only if* there exist $\delta > 0$, and α, $0 < \alpha < 1$, such that $\lambda(E) < \delta$ implies $\lambda(f^{-k}(E)) < \alpha$ for all $k \geqq 0$.

In this note we consider nonsingular maps of a finite measure space into itself and give a necessary and sufficient condition for the existence of an invariant measure which is absolutely continuous with respect to the given measure. The condition says, intuitively, that the iterated inverse images of "small" sets must not become too "large". The precise formulation is given in the theorem below. The problem of existence of invariant, absolutely continuous measures arises, for example, in the study of dynamical properties of interval maps; see [1] and [2], where one finds also further references. In a different direction, we note that the investigation of invariant measures, related to the given measure in terms of their null sets, started after the discovery of the individual ergodic theorem, since only zero values of the invariant measure enter into the conclusion of the theorem, compare [3].

A measure space $(\Omega, \mathfrak{B}, \lambda)$ is a triple such that $\mathfrak{B}$ is a σ-algebra of subsets of the set Ω, and λ is a measure (positive) defined on $\mathfrak{B}$. A measurable transformation f from Ω into itself is called *nonsingular* (with respect to λ) if

$$\lambda(A) = 0 \Rightarrow \lambda(f^{-1}(A)) = 0, \qquad \forall A \in \mathfrak{B}. \tag{1}$$

A set function ν on $\mathfrak{B}$ is called *f-invariant* if

$$\nu(A) = \nu(f^{-1}(A)), \qquad \forall A \in \mathfrak{B}. \tag{2}$$

We then have the following:

Theorem. *Let $(\Omega, \mathfrak{B}, \lambda)$ be a measure space with normalized measure λ, f a nonsingular transformation of Ω into itself. Then*

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