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Geometry of $SU(2)$ Gauge Fields

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Abstract. We study $SU(2)$ Yang-Mills theory on $S^3 \times \mathbb{R}$ from the canonical view-point. We use topological and differential geometric techniques, identifying the "true" configuration space as the base-space of a principal bundle with the gauge-group as structure group.

1. Introduction

We study in this paper the space of connections on the trivial $SU(2)$ bundle on S^3 and the action of the gauge-group on this space. Let $\mathscr{C} = \mathscr{C}^k$ denote the space of connections belonging to Sobolev class (k), $k \geqq 3$. We introduce the groups Aut, Aut^o (see Sect. 2) of gauge transformations belonging to the Sobolev class $(k+1)$. We then define the space $\mathscr{C}_o$ of generic connections, which are the connections whose holonomy coincides with the whole group $SU(2)$, and prove that the above groups act properly on $\mathscr{C}$ (Proposition 2.4) and that $\mathscr{C}_o$ in a principal Aut (or Aut^o) bundle (Propisition 4.3). The proof involves deriving estimates for certain elliptic operators whose coefficients belong to Sobolev spaces and are not necessarily C^∞. We define the groups Aut_e, Aut_e^o (Sect. 4b)) and show that the Aut (resp. Aut^o) bundle cannot be reduced to the subgroup Aut_e [resp. Aut_e^o (Theorem 5.1)]. In particular gauge-fixing is not possible. This result is proved by looking at left-invariant differential forms on $S^3 = SU(2)$ with values in the Lie algebra of $SU(2)$ and by showing essentially that the principal $SO(3)$ bundle obtained by the action on 3×3 real matrices of rank $\geqq 2$, by multiplication on the left, is nontrivial (Theorem 6.2).

In Sect. 7 we introduce the Coulomb connection. We show (Theorem 7.5) that, in case we use the biinvariant metric on $S^3 = SU(2)$, the values of the curvature form of this connection at the point $\omega/2 \in \mathscr{C}_o$, where ω is the Maurer-Cartan form, span a dense subspace in the gauge algebra.

The study was motivated by the following physical considerations, taking Dirac's theory [1] of singular Lagrangians as starting point. We may recall that the Faddeev-Popov procedure was derived [2] by an extension of Dirac's

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