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# Generalized Quantum Spins, Coherent States, and Lieb Inequalities*

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**Abstract.** A mathematical generalization of the concept of quantum spin is constructed in which the role of the symmetry group $O_3$ is replaced by $O_\nu$ ($\nu = 2, 3, 4, \ldots$). The notion of spin direction is replaced by a point on the manifold of oriented planes in $\mathbb{R}^\nu$. The theory of coherent states is developed, and it is shown that the natural generalizations of Lieb's formulae connecting quantum spins and classical configuration space hold true. This leads to the Lieb inequalities [1] and with it to the limit theorems as the quantum spin $l$ approaches infinity. The critical step in the proofs is the validity of the appropriate generalization of the Wigner-Eckart theorem.

## 1. Introduction

The study of the classical limit of certain quantum mechanical systems was the subject of an interesting paper of Lieb [1]. Among the systems considered there is the Heisenberg model whose Hamiltonian is of the form

$$H_{\text{quant}} = \sum g_{jk} \mathbf{L}^{(j)} \cdot \mathbf{L}^{(k)}, \tag{1.1}$$

where the $\mathbf{L}^{(j)}$ are independent quantum spins with the same total spin quantum number $l$. To this model corresponds a classical analogue, defined formally by the same energy function with the $\mathbf{L}^{(j)}$ interpreted as vectors of length $l$ in 3-dimensional space. If one writes $\mathbf{L}^{(j)} = l\boldsymbol{\omega}^{(j)}$, the $\boldsymbol{\omega}^{(j)}$ unit vectors, the classical model is then defined by its energy function

$$H_{\text{class}} = l^2 \sum g_{jk} \boldsymbol{\omega}^{(j)} \cdot \boldsymbol{\omega}^{(k)}. \tag{1.2}$$

From the point of view of statistical mechanics one is interested in the respective partition functions

$$Z_{\text{quant}}(l) = \frac{1}{(2l+1)^n} \operatorname{Tr} \exp(-H_{\text{quant}}) \tag{1.3}$$

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