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Extremal Λ-Inequalities for Ising Models with Pair Interactions

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Abstract. The inequalities for spin correlation functions of ferromagnetic Ising models with pair interactions derived in a previous paper are studied in more detail. It is shown that each of these inequalities is a positive linear combination of a finite number of "extremal" inequalities, which can in principle be determined and of which a number of examples is given.

1. Introduction

In a recent paper [1], to be referred to as I, two classes of relations between spin correlation functions of Ising models with pair interactions were studied. One of these classes consists of correlation-function inequalities, for ferromagnetic Ising models, of the type $\sum_{B\subset A} \lambda_B\langle\sigma_B\rangle\langle\sigma_B\sigma_D\rangle \geqq 0$, where A is an arbitrary set of spins of the system, D a subset of A, and $\{\lambda_B\}_{B\subset A}$ a set of real numbers which are independent of the coupling parameters of the system. In this paper this class of correlation-function inequalities (called Λ-inequalities) will be studied in more detail. In particular, we shall show that for every set of spins A with $|A|$ even there is a unique finite set of Λ-inequalities with $D=A$ from which all other Λ-inequalities with $D=A$ which are generally valid (i.e. valid for all Ising models containing the set A) can be derived by taking positive linear combinations. The method by which these extremal Λ-inequalities can, at least in principle, be found will be sketched. Examples of extremal Λ-inequalities valid for arbitrary sets A will be derived, and for the cases $|A|=4$ and $|A|=6$ all extremal Λ-inequalities will be given. The generalization to the more general case $D\subset A$ will form the subject of a subsequent paper.

2. Definitions and Notation

As in I, a *graph* G is defined as a pair $(V(G), E(G))$, where $V(G)$ is a set of elements called vertices and $E(G)$ a set of unordered pairs $\{v, v'\}$ of distinct vertices, called edges. G is *finite* if $V(G)$ and $E(G)$ are finite.

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