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The Klein-Gordon Equation with Light-Cone Data

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Abstract. It is shown that the characteristic Cauchy problem $\left(\frac{\partial^2}{\partial t^2} - \Delta + 1\right) \cdot u(x,t) = 0$, $u(x, -|x|) = f(x)$, $x \in \mathbb{R}^n$, $n \geqq 1$ has a unique finite energy weak solution for all f such that $\int dx(|\nabla f|^2 + |f|^2) < \infty$ and all finite energy weak solutions of the equation are obtained in this way.

1. Introduction

We shall consider the characteristic Cauchy problem for the Klein-Gordon $(K-G)$ equation

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta + 1\right) u(x,t) = 0\,, & (1.1) \\ u(x, -|x|) = f(x)\,, & (1.2) \end{cases}$$

where $x \in \mathbb{R}^n$, $n \geqq 1$ and $t \in \mathbb{R}$. We shall prove that this problem has a unique finite energy weak solution for all f such that $\int\limits_{\mathbb{R}^n} dx(|\nabla f|^2 + |f|^2) < \infty$, and all finite energy weak solutions of (1.1) are obtained in this way. In fact the energy E fulfills

$$E \equiv 1/2 \int\limits_{\mathbb{R}^n} dx \left(\left| \frac{\partial u}{\partial t}(x,t) \right|^2 + |\nabla u(x,t)|^2 + |u(x,t)|^2 \right)$$

$$= 1/2 \int\limits_{\mathbb{R}^n} dx(|\nabla f|^2 + |f|^2)\,. \qquad (1.3)$$

We shall also give an explicit formula for $u(x,t)$ in terms of its light-cone restriction f with the help of a "light-cone Fourier transform".

For some general results on characteristic Cauchy-problems see Hörmander [1]. The wave-equation has been considered by Riesz [2] and Strichartz [3].

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