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Remarks on the Modular Operator and Local Observables

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Abstract. In this paper we give a characterization of the modular group of a von Neumann algebra $\mathscr{R}$, with a cyclic and separating vector, which provides at the same time a necessary and sufficient condition so that two von Neumann algebras $\mathscr{R}_1$ and $\mathscr{R}_2$, such that $\mathscr{R}_1 \subseteq \mathscr{R}'_2$, are the mutual commutants, i.e. $\mathscr{R}_1 = \mathscr{R}'_2$.

An application is made to the duality property in Quantum Field Theory, and we give a sufficient condition for PCT invariance in a theory of local observables.

1. Introduction

It is known that if $\mathscr{R}$ is a von Neumann algebra with a cyclic and separating vector Ω, then the associated modular operator is characterized by the following conditions:

i) $\Delta = \Delta^*$, $\Delta > 0$;
ii) for each $t \in \mathbb{R}$ $\Delta^{it}\Omega = \Omega$;
iii) for each $t \in \mathbb{R}$ $\Delta^{it}\mathscr{R}\Delta^{-it} = \mathscr{R}$;
iv) the automorphism group $\sigma_t = \Delta^{it} \cdot \Delta^{-it}$, satisfies the KMS condition for the state $\omega_0 = (\Omega, \cdot\,\Omega)$.

Recall that $\Delta^{1/2}$ is the modulus in the polar decomposition of the *-operator $A\Omega \to A^*\Omega$, $A \in \mathscr{R}$; the phase J is an antiunitary involution such that $J\Delta^{1/2}A\Omega = A^*\Omega$, and $J\mathscr{R}J = \mathscr{R}'$. By these relations $\Delta^{1/2}\mathscr{R}^{\mathrm{sa}}\Omega = \mathscr{R}'^{\mathrm{sa}}\Omega$, where we denote with $\mathscr{R}^{\mathrm{sa}}$ the selfadjoint operators of $\mathscr{R}$ [8].

Conversely the KMS condition is easily implied by the condition

iv') $\Delta^{1/2}\mathscr{R}^{\mathrm{sa}}\Omega \subset \mathscr{R}'^{\mathrm{sa}}\Omega$.

In this note we show that condition iv') independently from Tomita-Takesaki theory, implies a commutation theorem, and at the same time characterizes the modular group, producing another proof of the uniqueness of the modular automorphisms.

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