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$\mathbb{C}P^2$ as a Gravitational Instanton

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Abstract. We compare some of the properties of $\mathbb{C}P^2$ with those of the SU(2) Yang-Mills Instanton and conclude that $\mathbb{C}P^2$ may be regarded as a gravitational pseudoparticle surrounded by an event horizon.

1. Introduction

This paper is one of three concerned with Riemannian solutions of the Einstein equations with cosmological constant Λ,

$$R_{\alpha\beta}-\tfrac{1}{2}Rg_{\alpha\beta}+\Lambda g_{\alpha\beta}=0. \tag{1}$$

The first [1] contains the general theory of such spaces and their role in quantum gravity. The second (this paper) treats a particular example, $\mathbb{C}P^2$. The third [2] deals with generalized spin structures in Riemannian spaces, taking $\mathbb{C}P^2$ as a particular example.

$\mathbb{C}P^2$ is a two dimensional complex manifold which may also be given a Riemannian metric (known to mathematicians as the Fubini-Study metric) which satisfies (1). The fact that $\mathbb{C}P^2$ has non-vanishing Pontrjagin number has led Eguchi and Freund [3] to consider $\mathbb{C}P^2$ as an analogue of the well known "Instanton" solution of the SU(2) Yang-Mills equations [4]. What one calls an instanton outside the domain of SU(2) Yang-Mills theory depends upon which features of the Yang-Mills solutions one is making an analogy with. In this paper we shall point out some of the similarities and the differences between the two cases and relate them to the general discussion of [1]. Before doing so (in Section 6) we shall collect together some properties of $\mathbb{C}P^2$. Most of these are well known to mathematicians but less well known in the physics community. Section 2 contains an account of $\mathbb{C}P^2$ as a complex manifold, together with its standard Kähler structure. In Section 3 we discuss the isometry group ($SU(3)/Z_3$) and a particular 4-dimensional subgroup. The possession of a 4-dimensional isometry group acting on 3-spheres is characteristic of the Taub-NUT family of solutions of the Einstein equations and we show $\mathbb{C}P^2$ to be a limiting case of the general form.