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# Amenability of Crossed Products of $C^*$-Algebras

Jonathan Rosenberg*

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19174, USA

**Abstract.** It is shown that the class of amenable (resp. strongly amenable) $C^*$-algebras is closed under the process of taking crossed products with discrete amenable groups. Under certain circumstances, amenability is also preserved under taking a "crossed product" with an amenable semigroup of linear endomorphisms. These facts are used to show that certain simple $C^*$-algebras $\mathcal{O}_n$ studied by J. Cuntz are amenable but not strongly amenable (thus answering a question of B. E. Johnson), yet are stably isomorphic to strongly amenable algebras.

The purpose of this note is to prove stability of the classes of amenable and strongly amenable $C^*$-algebras under the process of taking crossed products with discrete amenable groups (and under certain circumstances, with discrete amenable semigroups). These facts are used to show that certain simple $C^*$-algebras $\mathcal{O}_n$, studied by Cuntz in [4], are amenable but not strongly amenable, yet are stably isomorphic to strongly amenable algebras. We thus answer a question of Johnson [7], who wondered if every amenable $C^*$-algebra is strongly amenable, and at the same time show that strong amenability is not preserved under "cutting down" by projections. The questions of whether amenability is a stable isomorphism invariant, and whether every nuclear $C^*$-algebra is amenable, remain open, and the example of the algebras $\mathcal{O}_n$ seems to indicate that positive answers to these are at least plausible. The present results also show that a strongly amenable $C^*$-algebra need not be an inductive limit of Type I algebras.

We use the following notation. If $X$ is a Banach space, $X^*$ denotes its dual. For $K \subseteq X^*$, $\overline{\text{co}}\,K$ denotes its weak-* closed convex hull. If $A$ is a $C^*$-algebra with unit, $U(A)$ denotes the unitary group of $A$. If $A$ is any $C^*$-algebra, we use the notation $A^\sim$ to mean $A$ if $A$ has a unit, $A$ with identity adjoined otherwise. Recall that a $C^*$-algebra $A$ is *amenable* if and only if for every 2-sided unital Banach $A^\sim$-module $X$ and for every continuous derivation $d: A \to X^*$, $d$ is the coboundary of some element of $X^*$ [7, p. 61]. (One may drop the continuity assumption on $d$ since, by [8], every

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