Commun. math. Phys. 56, 195—212 (1977)

Communications in
Mathematical
Physics

Instantons in a $U(1)$ Lattice Gauge Theory: A Coulomb Dipole Gas

James Glimm*
Rockefeller University, New York, NY 10021, USA

Arthur Jaffe**
Harvard University, Cambridge, MA 02138, USA

Abstract. We study the decomposition $A = A_I + A_{SW}$ of a $U(1)$ lattice gauge field into instanton and spin wave parts. The action also decomposes, $\mathscr{A} = \mathscr{A}_I + \mathscr{A}_{SW} + R$. Here $\mathscr{A}_I$ is a Coulomb dipole gas, $\mathscr{A}_{SW}$ is a zero mass free field, and R is a higher order remainder. We study $\mathscr{A}_I$ in detail, for $d \geqq 4$, in the dilute gas case (which corresponds to the low temperature limit of the gauge field theory). We establish the leading behavior of the free energy: $f \sim \varepsilon^{-d} a \zeta$. Here ε is the lattice spacing, a is a geometrical constant and ζ is an activity defined in terms of a small number of instanton configurations. Our methods suggest the absence of screening in the dilute dipole gas, $d \geqq 4$, in contrast to Debye screening for the dilute monopole gas.

1. Introduction

It has been proposed by Gell-Mann and others that a gauge field coupled to a fermion (quark) field may describe the internal structure of protons, neutrons, etc., as quark triplets, etc. In order to account for the strong binding of individual quarks, as well as the observed property of asymptotic freedom of the physical particles, a qualitative understanding of the phase transitions and critical points of the gauge-quark system appears necessary [11]. The important properties of the critical points are inferred from renormalization group and semiclassical methods.

In this paper we study some steps in this program on a rigorous mathematical level. See [8] for an introduction to the mathematics of lattice gauge fields. Following Polyakov [9], we formally derive the instanton interaction in terms of a dilute Coulomb gas for a $U(1)$ lattice gauge theory. For $d \geqq 4$, the gas consists of vector charges, and a conservation law restricts the charges to occur only in dipoles. Since the dipole-dipole interaction is not integrable at long distances, standard methods do not apply. Nevertheless, we prove an upper and lower bound on the free

* Supported in part by the National Science Foundation under Grant PHY 76-17191
** Supported in part by the National Science Foundation under Grant PHY 75-21212