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Dirichlet Forms and Markov Semigroups on C^*-Algebras*

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Abstract. We extend the classical theory of Dirichlet forms and associated Markov semigroups to the case of a C^*-algebra with a trace. Semigroups of completely positive maps are characterized by completely positive Dirichlet forms.

1. Introduction

A powerful method for the generation of Markov processes in the commutative case is given by the classical theory of Dirichlet forms and spaces. This theory has its roots in classical potential theory and has been developed particularly since the fundamental work of Beurling and Deny [6]. The theory is closely related with Dynkin's and Hunt's theory of stong Markov processes and has been greatly developed recently in its symmetric L_2-version particularly by Fukushima and Silverstein, see [20, 21, 30, 31] and [2–4].

Since the theory of Dirichlet forms in the commutative case deals with forms which are monotone with respect to a class of contractions applied to certain subalgebras of continuous functions, it is natural to expect a non commutative extension of the theory to the case of C^*-algebras. It is the purpose of this paper to show that, at least in the case of C^*-algebras with a trace, this idea can actually be carried through. The outcome are Markov semigroups, i.e. positivity preserving semigroups of maps, and completely Markov semigroups, i.e. semigroup of completely positive maps. Positive and completely positive maps of C^*-algebras have been the object of many investigations, standard references for foundational work are e.g. [5, 8, 32, 33]. More recently a considerable renewed interest in completely positive maps has arisen particularly in connection with certain foundational problems of non equilibrium statistical mechanics. We allude here to the large body of work on the so called quantum dynamical semigroups and quantum stochastic process, see e.g. [1, 7, 9, 10, 12–19, 22–28]. Notably a classification of norm continuous completely positive map on the C^*-algebra $\mathcal{B}(\mathcal{H})$ bounded operators on a Hilbert space has been achieved, [28], see also [22]. For

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