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Pure Soliton Solutions of Some Nonlinear Partial Differential Equations

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Abstract. A general approach is given to obtain the system of ordinary differential equations which determines the pure soliton solutions for the class of generalized Korteweg-de Vries equations (cf. [6]). This approach also leads to a system of ordinary differential equations for the pure soliton solutions of the sine-Gordon equation.

§ 1

A numerical study [10] of the Korteweg-de Vries equation (KdV in short)

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

showed that some solutions of this nonlinear partial differential equation decompose for large time t into solitary waves travelling with constant velocity (the so called solitons). A better understanding of this phenomenon was possible when Gardner et al. [5] developed their ingenious Schrödinger operator calculus which gave rise to the inverse scattering method for solving the KdV. Since then similar methods have been developed for other nonlinear partial differential equations ([1], [13], [19]).

In a fundamental paper [12] Lax gave a better explanation of this phenomenon by considering the gradients of the conservation laws of the KdV equation. As a result of this he gave a precise description of the 2-soliton case. An integral equation (Gelfand-Levitan eq.) for the N-soliton case was already provided by the inverse scattering method. Considering explicit solutions of this integral equation the authors of [6] obtained a system of ordinary differential equations for the N-soliton solution of the KdV. One advantage of this method is that it permits identification of solitons even during the interaction process.

In this paper a general approach to this system of ordinary differential equations is given. The only tool on which our method depends is the time invariance of the spectrum of a family of Schrödinger operators. No other spectral properties are used and we do not work with reflection and normalization coefficients. Therefore we have also found a description of the N-soliton case for the