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Hausdorff Measure and the Navier-Stokes Equations

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Abstract. Solutions to the Navier-Stokes equations are continuous except for a closed set whose Hausdorff dimension does not exceed two.

1. Informal Statement of Results

Let $v:R^3 \to R^3$ be a divergence free, square integrable vector field on 3-space. We will show that there exists a function $u:R^3 \times R^+ \to R^3$ ($R^+ = \{t:t>0\}$ is time) which is a weak solution to the Navier-Stokes equations of incompressible fluid flow with viscosity $=1$ and initial conditions v, and which satisfies the following: There exists a set $S \subset R^3 \times R^+$ such that the two dimensional Hausdorff measure of S is finite, $(R^3 \times R^+) - S$ is an open set, and the restriction of u to $(R^3 \times R^+) - S$ is a continuous function.

The above will be derived as a consequence of a more general theorem in which u satisfies a weak form of the Navier-Stokes equations with an external force $f:R^3 \times R^+ \to R^3$ which is divergence free with the property $f(x,t) \cdot u(x,t) \leqq 0$.

2. Notation and Complete Statement of Results

Hausdorff measure is defined in [2, p. 171]. We set $R^+ = \{t \in R : t > 0\}$ and $B(a,r) = \{x \in R^3 : |x-a| \leqq r\}$ for all $a \in R^3$ and $r>0$. The norm $|\ |$ is always euclidean norm and $\|\ \|_p$ is the L^p norm. Open and closed intervals are denoted (a,b) and $[a,b]$, respectively. If $f:X \to R$ and $A \subset X$ then $\sup(f,A)$ is the supremum of f over A and $\operatorname{spt}(f)$ is the closure of $\{x : f(x) \neq 0\}$. If f and g are functions defined on a subset of $R^3 \times R$, h is a function on R^3, and k is a function on R, then we set

$$(f*g)(x,t) = \iint f(y,s)g(x-y,t-s)dyds,$$
$$(f*h)(x,t) = \int f(y,t)h(x-y)dy,$$
$$(f*k)(x,t) = \int f(x,s)k(t-s)ds$$

whenever the integrals make sense. If $X = R^3$, $X = R$, or $X = R^3 \times R^+$, we let $C^\infty(X,R)$ be the set of infinitely differentiable functions $f:X \to R$. In addition, $C_0^\infty(X,R)$ is the