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The ε-Expansion for the Hierarchical Model

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Abstract. In this paper, the Hierarchical Model is studied near a non-trivial fixed point φ_ε of its renormalization group. Our analysis is an extension of work of Bleher and Sinai. We prove the validity of the ε-expansion for φ_ε. We then show that the renormalization transformations around φ_ε have an unstable manifold which is completely characterized by the tangent map and can be brought to normal form. We then establish relations between this result and the critical behaviour of the model in the thermodynamic limit.

Introduction and Description of Results

This paper brings the ε-expansion of the renormalization group theory for the Hierarchical Model on a sound mathematical footing. The Hierarchical Model is a model on a one-dimensional lattice with ferromagnetic spin interaction whose range depends on a parameter c. As c varies, the behaviour of the model near its critical temperature varies also and actually multicritical points of any degree can occur. The first non-Gaussian critical behaviour occurs when $c=2^{1/2(1-\varepsilon)}$ and then the fixed point of the renormalization group, (which is an exact transformation for this model) can be discussed by the so-called ε-expansion. This model is the simplest model in which an ε-expansion arises [1, 2, 7–9]. The main impetus for the mathematical study of this model comes from the deep work of Bleher and Sinai [3, 4], on which we rely for the existence of a critical spin distribution.

In Section 1, we review the definition and the exact meaning of the ε-expansion for this model (one changes the range of the interaction instead of the dimension). We show that the ε-expansion is the perturbation theory of bifurcation from a simple eigenvalue [5].

Section 2 is the basis of all our results on the validity of the ε-expansion; we show that the fixed point of the renormalization group is differentiable in ε up to any order, provided ε is sufficiently small, and has thus an ε-expansion up to any order. The proofs take up Sections 2–5.

Section 6 is the description of the renormalization group action near the fixed point; this is the theory of the normal form of diffeomorphisms around a fixed point (on Banach spaces [17]).