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Foliations of Space-Times by Spacelike Hypersurfaces of Constant Mean Curvature

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Abstract. The foliations under discussion are of two different types, although in each case the leaves are C^2 spacelike hypersurfaces of constant mean curvature. For manifolds, such as that of the Friedmann universe with closed spatial sections, which are topologically $I \times S^3$, I an open interval, the leaves will be spacelike hypersurfaces without boundary and the foliation will fill the manifold. In the case of the domain of dependence of a spacelike hypersurface, S, with boundary B, the leaves will be spacelike hypersurfaces with boundary, B, and the foliation will fill $D(S)$.

It is shown that a local energy condition ensures that the constant mean curvature increases monotonically with time through such foliations and that, in the case of a foliation whose leaves are spacelike hypersurfaces without boundary in a manifold where this energy condition is satisfied globally, the foliation is unique.

In the Robertson-Walker cosmologies there is a geometrically preferred time coordinate, namely the mean curvature of the isotropic spacelike sections which foliate the manifolds. It is the purpose of the following to show that this property of the mean curvature of the leaves of such a foliation holds true in more general circumstances. The more difficult question of the existence of such foliations will not be dealt with in this paper. It seems likely that, although it presumably fails in general, it will hold in many cases of interest. The following (restricted) definitions will be adopted.

Definition 1. A space-time is a four dimensional, C^∞, pseudo-Riemannian, time-orientable manifold of signature $(+---)$.

Definition 2. A C^k foliation, F, of a manifold, M, by spacelike hypersurfaces without boundary is a map $F: M \to I$, where I is an interval, with the properties

(i) F is a C^k map.

(ii) F is onto.

(iii) If $t \in I$, then $F^{-1}(t)$ is a C^k spacelike hypersurface without boundary. $F^{-1}(t)$ is called a leaf of the foliation F.