Commun. math. Phys. 54, 237—248 (1977)

Communications in
Mathematical
Physics

A Theorem of Šarkovskii on the Existence of Periodic Orbits of Continuous Endomorphisms of the Real Line

P. Štefan

Institut des Hautes Etudes Scientifiques, F-91440 Bures-sur-Yvette, France

Abstract. Two theorems are proved—the first and the more important of them due to Šarkovskii—providing complete and surprisingly simple answers to the following two questions: (i) given that a continuous map T of an interval into itself (more generally, into the real line) has a periodic orbit of period n, which other integers must occur as periods of the periodic orbits of T? (ii) given that n is the least odd integer which occurs as a period of a periodic orbit of T, what is the "shape" of that orbit relative to its natural ordering as a finite subset of the real line? As an application, we obtain improved lower bounds for the topological entropy of T.

Consider an order relation $\vdash$ on the set N of all integers $\geqq 1$, defined as follows. Let $N = A \cup B$, $A = \{2^n l : n \leqq 0, l \leqq 3, l \text{ odd}\}$, and $B = \{2^m : m \leqq 0\}$. Order A lexicographically with increasing n and l; order B with decreasing m, and let A precede B. We have

$$3 \vdash 5 \vdash 7 \vdash 9 \vdash \ldots \vdash 2 \cdot 3 \vdash 2 \cdot 5 \vdash \ldots \vdash 4 \cdot 3 \vdash \ldots \vdash 8 \vdash 4 \vdash 2 \vdash 1 .$$

The main result of [1] is

Theorem 1 (Šarkovskii). *Let $T: \mathbb{R} \to \mathbb{R}$ be a continuous mapping which has a periodic orbit of period n. Then T has a periodic orbit of period m for every $m \in N$ such that $n \vdash m$.*

The main aim of these notes is to make the contents of [1] available to those who do not read Russian. The reader should be warned that this is not a translation: some new results, closely related to Šarkovskii's work, are presented in Sections E and H, and the material of [1] has been rearranged and modified to suit my taste and to avoid one or two mistakes which have crept into Šarkovskii's argument. Nonetheless, I believe that all the main points of [1] and here, and I have tried not to omit anything potentially useful.

The proof of Theorem 1 occupies Sections A–D below. Section E contains the proof of the fact that the "minimal" odd orbits are, up to an order preserving or order reversing isomorphism, uniquely determined by their period. This result