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The Euclidean Loop Expansion for Massive $\lambda\Phi_4^4$: Through One Loop

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Abstract. As an application of the theory of solutions of the classical, Euclidean field equation, we prove the existence of solutions to the renormalized functional field equation, for the $\lambda\Phi^4$ interaction in four Euclidean space dimensions, with non-negative λ and nonzero mass, through order $\hbar c$. That is, we prove that the functional derivative of the connected generating functional is in the Schwartz space $\mathrm{Re}\mathscr{S}(R^4)$, when evaluated at external sources in $\mathrm{Re}\mathscr{S}$, through order $\hbar c$. We also prove the existence of all functional derivatives of the connected generating functional through the same order. All quantities of interest are analytic in the coupling constant at $0 \leqq \lambda < \infty$, and continuous in the external source.

I. Introduction

A large number of formal, and several exact results, already exist for the loop expansion of the generating functional for connected, time-ordered vacuum expectation values of scalar field operators over Minkowski space. In this paper, we begin to develop the Euclidean version of the loop expansion for the massive scalar field with $\lambda\Phi^4$ interaction, $\lambda \geqq 0$, in four Euclidean dimensions, by proving the existence of the renormalized theory through order $\hbar c$ (one loop). We do that by studying the functional form of the renormalized Euclidean field equation. The techniques of linear and nonlinear functional analysis have matured to the point where this becomes a "standard" calculation, and we think it reasonable to hope that the same is true to all orders in the loop expansion.

Some motivating remarks follow:

(i) In the Minkowski version, Jackiw [1] gives a systematic treatment of the effective potential in the loop expansion, and he discusses the renormalization of one and two loops for $\lambda(\Phi^4)_{1+3}$ in some detail. We are interested in the Euclidean version because it is somewhat easier to state and prove rigorous theorems. We say "somewhat", because the classical field equation in the presence of an external source plays a central role; and the mathematics for the classical field equation in the Minkowski $\lambda\Phi^4$ theory is well developed, albeit in the absence of external