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Relative Entropy and the Wigner-Yanase-Dyson-Lieb Concavity in an Interpolation Theory

A. Uhlmann

Department of Physics and NTZ, Karl-Marx-University,
DDR-701 Leipzig, German Democratic Republic

Abstract. We show that the Wigner-Yanase-Dyson-Lieb concavity is a general property of an interpolation theory which works between pairs of (hilbertian) seminorms. As an application, the theory extends the relevant work of Lieb and Araki to positive linear forms of arbitrary *-algebras. In this context a "relative entropy" is defined for every pair of positive linear forms of a *-algebra with identity. For this generalized relative entropy its joint convexity and its decreasing under identity-preserving completely positive maps is proved.

1. Introduction

In this note we establish a generalization of the important Wigner-Yanase-Dyson conjecture that was proved by Lieb [1]. More precisely, we describe an interpolation procedure having the WYDL-concavity property. Essentially, the WYDL-concavity is the joint concavity in A, B of

$$A, B \to \mathrm{Tr}\,(X^* A^{1-t} X B^t)\,, \qquad 0 < t < 1\,, \tag{1}$$

whenever (1) is well defined. The derivation of (1) at the value $t=0$ leads to the joint concavity of the expression

$$A, B \to \mathrm{Tr}\,(X^* A X \ln B) - \mathrm{Tr}\,(X X^* A \ln A)\,. \tag{2}$$

This functional reduces up to the sighn for $X = \mathbf{1}$ to the relative entropy of Lindblad [2], see also Umegaki [3]. From the concavity of (2) there is a short way to prove the strong subadditivity of entropy [4].

With the aid of the Tomita-Takesaki theory, Araki [5] was able to generalize most of the known properties of (1) and of the relative entropy to pairs of faithful normal states of W^*-algebras. In Araki's theory one has to rewrite (1) with the help of the relative modular operator of the two positive functionals

$$Y \to \mathrm{Tr}\,(YA) \quad \text{and} \quad Y \to \mathrm{Tr}\,(YB) \tag{3}$$