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Spectral Theory of the Operator $(p^2+m^2)^{1/2}—Ze^2/r$

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Abstract. Using dilation invariance and dilation analytic techniques, and with the help of a new virial theorem, we give a detailed description of the spectral properties of the operator $(\boldsymbol{p}^2+m^2)^{1/2}-Ze^2/r$. In the process the norm of the operator $|\boldsymbol{x}|^{-\alpha}|\boldsymbol{p}|^{-\alpha}$ is calculated explicitly in $L^p(\mathbb{R}^N)$.

I. Introduction

The *classical* Hamiltonian describing the interaction of a relativistic particle of charge e and mass m with an electromagnetic field [vector potential $\boldsymbol{A}(\boldsymbol{x})$ and scalar potential $\phi(\boldsymbol{x})$] is given by [1]

$$[(\boldsymbol{p}-e\boldsymbol{A}(\boldsymbol{x}))^2+m^2]^{1/2}+e\phi(\boldsymbol{x}). \tag{1.1}$$

To make the transition to quantum mechanics, the usual procedure (which is of course fraught with ambiguities) is to change the classical Hamiltonian into an operator on the Hilbert space $L^2(\mathbb{R}^3)$ by replacing $\boldsymbol{p}$ by $-i\nabla$. Because of the troublesome square root in (1.1), the standard procedure just described has received very little attention in treating a relativistic particle in an electromagnetic field. Historically, an alternative procedure was followed resulting in the Klein-Gordon (K.G.) equation [2]. Calling the energy function of (1.1) E, one finds

$$(E-e\phi(\boldsymbol{x}))^2-(\boldsymbol{p}-e\boldsymbol{A}(\boldsymbol{x}))^2-m^2=0.$$

One now makes the Ansatz $\boldsymbol{p}=-\mathrm{i}\nabla$ and tries to solve the implicit eigenvalue problem

$$\{(E-e\phi(\boldsymbol{x}))^2-(\boldsymbol{p}-e\boldsymbol{A}(\boldsymbol{x}))^2-m^2\}\psi(\boldsymbol{x})=0 \tag{1.2}$$

subject to "appropriate" boundary conditions. The K.G. equation has a definite virtue when the interaction is the Coulomb potential ($\boldsymbol{A}\equiv 0, \phi(\boldsymbol{x})=-Ze/|\boldsymbol{x}|$): The equation can be solved explicitly. It seems to us that this explicit solvability is the

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