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Convergence of Operator Product Expansions on the Vacuum in Conformal Invariant Quantum Field Theory

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Abstract. In a conformal invariant quantum field theory (in 4 space time dimensions) Wilson operator product expansions converge on the vacuum, because they are closely related to conformal partial wave expansions.

1. Introduction

Let $\phi^i(x)$, $\phi^j(y)$ two local quantum fields. According to Wilson [1], their product should admit an asymptotic expansion at short distances of the form

$$\phi^i(\tfrac{1}{2}x)\phi^j(-\tfrac{1}{2}x)\Omega=\sum_k C^{ijk}(x)\phi^k(0)\Omega\,. \tag{1.1a}$$

Herein ϕ^k are local fields, and $C^{ijk}(x)$ are singular c-number functions. In a scale invariant theory they are homogeneous functions of x. The expansion is presumably valid for all states Ω in the field theoretic domain $\mathscr{D}$ which is created out of the vacuum by polynomials in smeared field operators. We shall however only consider the special case

$$\Omega=\text{vacuum}\,. \tag{1.1b}$$

Studies in perturbation theory [2] indicate that expansion (1.1) is then valid as an asymptotic expansion to arbitrary accuracy for matrix elements $(\Psi, \phi^i(x)\phi^j(y)\Omega)$, Ψ in $\mathscr{D}$. This means that the error in a truncated expansion can be made smaller than any given power of x at sufficiently small distances $\|x\|$ by taking into account sufficiently many terms. (For more precise formulation cp. e.g. Appendix A of [3].)

Asymptotic expansions need not converge. For instance the asymptotic expansion near $y=0$ of the function $f(y)=\exp(-1/y)$ of one positive real variable y in powers of y vanishes identically and does therefore not converge to the function f.

Among the fields ϕ^k there are derivatives of other local fields. In general there appears $\partial^\mu\phi$ etc. together with any nonderivative field ϕ. In a conformal invariant theory, non-derivative fields ϕ can be recognized by their conformal transformation law [4], viz. $[\phi(0), K^\mu]=0$, $K^\mu=$generators of special conformal transformations.