Commun. math. Phys. 50, 241—244 (1976)

Communications in Mathematical Physics

Enveloping Subspaces and Superposition of States

Vittorio Cantoni
Politecnico di Torino, Torino, Italy

Abstract. In the space of pure states of a generic physical system, a family $\mathscr{L}$ of subsets is singled out and used to extend the quantum-mechanical notion of "superposition" of pure states. $\mathscr{L}$ possesses a natural lattice structure and corresponds to the lattice of closed subspaces of Quantum Mechanics.

1. Introduction

We have shown in a previous paper [1] that for any physical system described by a set $\mathscr{S}$ of *states*, a set $\mathcal{O}$ of *observables* and a *probability function* $p(A, \alpha, E)$, (the probability that the measurement of the observable A on the state α give a result in the Borel set E of the real line R), one can define a function $T(\alpha, \beta)$, $(\alpha, \beta \in \mathscr{S})$, which is a generalization of the quantum-mechanical transition probability between pure states.

Let us consider again a physical system with the same degree of generality as assumed in [1], and denote by $\mathscr{S}_p$ the set of all its pure states. We shall show that every subset $\mathscr{U}$ of $\mathscr{S}_p$ determines a second subset $\tilde{\mathscr{U}}$ of $\mathscr{S}_p$ which will be called the *enveloping subspace* generated by $\mathscr{U}$ and corresponds, in the case of a quantum-mechanical system, to the closed subspace U, in the Hilbert space H of the theory, generated by the representatives of the states belonging to $\mathscr{U}$.

Each subspace $\tilde{\mathscr{U}}$ is associated with a real function $T^{\mathscr{U}}(\alpha, \beta)$ defined in $\mathscr{S}_p \times \mathscr{S}_p$. In Quantum Mechanics $T^{\mathscr{U}}(\alpha, \alpha)$ is related to the norm of the projection on U of the unit representatives of α.

As a consequence it is possible to extend to any physical system the quantum-mechanical notion of "superposition of pure states", together with an appropriate definition of the "relative phase coefficient" between distinct superpositions of pairs of orthogonal pure states.

It is remarked that the set of all the enveloping subspaces possesses a natural lattice structure, equivalent, in the case of a quantum-mechanical system, to the lattice of closed subspaces of H. Such a structure is obtained here with no explicit reference to the lattice of propositions [3, 4, 5, 6].