Commun. math. Phys. 50, 103—112 (1976)

Communications in
Mathematical
Physics

Quasi-free "Second Quantization"

Lars-Erik Lundberg
Matematisk Institut HCØ, Copenhagen University, DK-2100 Copenhagen, Denmark

Abstract. Araki and Wyss considered in 1964 a map $A \to Q(A)$ of one-particle trace-class observables on a complex Hilbert-space $\mathscr{H}$ into the fermion C^*-algebra $\mathfrak{A}(\mathscr{H})$ over $\mathscr{H}$. In particular they considered this mapping in a quasi-free representation.

We extend the map $A \to Q(A)$ in a quasi-free representation labelled by T, $0 \leqq T \leqq I$, to all $A \in B(\mathscr{H})_{sa}$ such that $\operatorname{tr}(TA(1-T)A) < \infty$ with $Q(A)$ now affiliated with the algebra. This generalizes some well-known results of Cook on the Fock-representation $T=0$.

1. Introduction

Let $\mathfrak{A}(\mathscr{H})$ denote the fermion C^*-algebra over a complex Hilbert space $\mathscr{H}$, i.e. there exists a conjugate linear mapping $f \mapsto a(f)$ of $\mathscr{H}$ into $\mathfrak{A}(\mathscr{H})$, whose range generates $\mathfrak{A}(\mathscr{H})$ as a C^*-algebra such that $a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle I$, $a(f)a(g) + a(g)a(f) = 0$ for all $f, g \in \mathscr{H}$ and where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathscr{H}$.

A gauge-invariant quasi-free state ω_T of $\mathfrak{A}(\mathscr{H})$ is uniquely defined by the n-point functions $\omega_T(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_m)) = \delta_{nm} \det(\langle g_i, Tf_j \rangle)$ where $T \in B(\mathscr{H})$ and $0 \leqq T \leqq I$. Denote by $\mathscr{H}_T$, π_T and Ω_T the Hilbert-space, the representation, and the cyclic unit-vector associated with ω_T via the GNS-construction, i.e. $\omega_T(x) = (\Omega_T, \pi_T(x)\Omega_T)$, $x \in \mathfrak{A}(\mathscr{H})$.

Let A be a self-adjoint (s.a.) finite-rank operator on $\mathscr{H}$, i.e. there exists an orthonormal set $\{u_n\}_{n=1}^N$ in $\mathscr{H}$ and $\alpha_n \in \mathbb{R}$ such that $Af = \sum_{n=1}^{N} \alpha_n u_n \langle u_n, f \rangle$ for $f \in \mathscr{H}$. Araki and Wyss [1] considered the following map Q of finite-rank s.a. operators on $\mathscr{H}$ into $\mathfrak{A}(\mathscr{H})_{sa}$, $A \overset{s}{\mapsto} Q(A) = \sum_{n=1}^{N} \alpha_n a(u_n)^* a(u_n)$, which has the following properties:

$$Q(A) + Q(B) = Q(A+B)\,, \tag{1.1}$$

$$[Q(A), a(f)^*] = a(Af)^*\,, \tag{1.2}$$

$$i[Q(A), Q(B)] = Q(i[A, B])\,. \tag{1.3}$$