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An Application of Morse Theory to Space-Time Geometry

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Abstract. Milnor's treatment [6] of Morse's global theory of the calculus of variations for geodesics [7] is restated in the context of space-time geometry: it is seen as providing a link between the curvature and the causal structure of a stably causal globally hyperbolic Lorentzian manifold. An application is discussed.

Introduction

Morse's global theory of the calculus of variations is the basis of a number of theorems relating the curvature and topology of Riemannian manifolds [6, 7]. In this paper I shall describe a method whereby the theory can be restated in the context of space-time geometry and discuss its potential usefulness in dealing with global problems in general relativity.

The first three sections of the paper are an outline of the principal physical and mathematical ideas involved, leading up to a statement of the main theorem at the end of § 3: these sections can be regarded as an extended introduction (more detailed accounts of some of the material covered can be found in [1, 3, 4, 6, 9]). A large proportion of the argument consists of adapting standard elementary results from algebraic topology and Riemannian geometry. In order to keep the paper reasonably selfcontained, I have given outlines of the concepts involved and sketched proofs of the theorems before describing the necessary (but, for the most part, trivial) modifications. Only where the argument diverges radically from that used in Riemannian geometry have I gone into the full technical details.

The fourth section is a proof of the theorem.

Notation. Throughout, M denotes a smooth (C^∞) paracompact Hausdorff manifold of dimension greater than two in which is given a causal Lorentzian metric $\boldsymbol{g}$ with signature $(+, -, -\ldots)$. This means that $(M, \boldsymbol{g})$ is time oriented (the two halves of the light cone are labelled continuously throughout M as future and