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Mathematics of Noncanonical Quantum Theory

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Abstract. The Weyl relation $e^{-isQ}e^{itP}e^{isQ}=e^{it(P+sI)}$ is generalized so as to hold for noncanonical couples (P, Q) implying the commutation relation $i[P,Q]=C$ where C is arbitrary bounded self-adjoint. It is shown that if 0 is not in the closure of the numerical range of C then both P and Q are spectrally continuous and neither bounded from below nor above. The dynamical equations in noncanonical theory are established. It is shown that H (which is no longer given by correspondence) cannot be bounded from below (above) if $C \leqq 0$ $(C \geqq 0)$, $C \neq 0$.

Introduction

Consider a one-dimensional possibly nonlinear quantum mechanical oscillator,

$$i[H,Q]=P\,, \tag{0.1a}$$

$$i[H,P]=-F(Q)\,, \tag{0.1b}$$

where P, Q, and H are the operators of momentum, position, and energy, respectively, and where F is a nice function, for example, a polynomial. In addition, P and Q satisfy the canonical commutation relation

$$i[P,Q]=I \quad (=\text{identity operator})\,. \tag{0.1c}$$

Some years ago it was suggested by Heisenberg (cf. [1]) to replace (0.1c) by a "noncanonical" commutation relation,

$$i[P,Q]=C\,, \tag{0.2}$$

where C is some self-adjoint linear operator. The motivation for this proposal has to do with the removal of divergencies in quantum field theory. But there are also other reasons to study noncanonical quantum theories, for example, general relativistic models. Clearly, H cannot be in correspondence with the classical Hamiltonian of (0.1a, b) if C is not a multiple of the identity operator. However, this does not matter as long as we can expect to have the operator H the desired spectral properties. In a paper to follow it will be shown that, for example, the dynamical equations (0.1a, b) have solutions for a large class of functions F (such