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Extremal Decomposition of Wightman Functions and of States on Nuclear *-Algebras by Choquet Theory

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Abstract. We give a short proof for the decomposability of states on nuclear *-algebras into extremal states by using the integral decompositions of Choquet and the nuclear spectral theorem, recovering a recent result by Borchers and Yngvason. The decomposition of Wightman fields into irreducible fields is a special case of this. We also indicate a quick solution of the moment problem on nuclear spaces.

Recently, Borchers and Yngvason [1] developed an extension theory for *-algebras of unbounded operators and applied it to the extremal decomposition of states[1] on nuclear *-algebras. The Choquet theory of extremal decompositions on cones [3] seemed to be not applicable; for the Borchers algebra $\mathscr{S}$ of test functions this has been discussed in [1] and [13]. In this paper we bypass the difficulty in a very simple way by going over to a larger cone to which Choquet theory can be applied[2] and then use nuclearity via the nuclear spectral theorem [5, 8].

Theorem [1]. *Let $\mathfrak{A}$ be a nuclear *-algebra with unit element, and let T be a state on $\mathfrak{A}$ such that $x \mapsto T(x^*x)$ is continuous*[3]. *Then there is a standard measure space*[4] *Z, a weakly measurable map $\zeta \to T_\zeta$ of Z to extremal states on $\mathfrak{A}$ and a positive measure ϱ on Z with $\varrho(Z)=1$ such that*

$$T = \int_Z T_\zeta d\varrho(\zeta) . \tag{1}$$

The main idea is to use the following observation.

Lemma. *Let $\mathfrak{A}_0$, a *-algebra with unit, be the finite linear span of a countable set of elements. Then $\mathfrak{A}^*_{0+}$, the positive cone in the algebraic dual $\mathfrak{A}^*_0$ equipped with the weak topology, is proper, metrizable and weakly complete.*

Proof. The topology of $\mathfrak{A}^*_0$ is given by a countable family of semi-norms and thus metrizable. Since $\mathfrak{A}^*_0$ contains *all* linear functionals on $\mathfrak{A}_0$, $\mathfrak{A}^*_0$ and $\mathfrak{A}^*_{0+}$ are weakly complete. $\mathfrak{A}^*_{0+}$ is proper since $\mathfrak{A}_{0+}$ contains the unit. QED.

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[1] States are positive continuous linear functionals which are 1 on the unit element.

[2] A similar idea has been used before by the author in [6, 7].

[3] This means that the associated representation is strongly continuous. In most cases this is automatically implied by the continuity of T, e.g. for the Borchers algebra $\mathscr{S}$ of test functions.

[4] One can assume $Z=[0, 1]$; cf. [4], B 20.