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A Remark on a Theorem of Powers and Sakai

S. Doplicher

Istituto Matematico "G. Castelnuovo", Università di Roma, I-00185 Roma, Italy

Abstract. Given an abelian locally compact group G and a C^*-algebra with unit $\mathfrak{A}$, the set of those continuous representations of G by automorphisms of $\mathfrak{A}$ which fulfill a spectrum condition is closed.

In a recent paper [1] Powers and Sakai proved, among other things, that if a sequence of continuous one-parameter groups of automorphisms of a C^*-algebra with identy, each with a generator in the algebra, converges strongly, uniformly on compact sets of the line, the limit one parameter group has a ground state.

As any one parameter group with a generator in the algebra has a ground state [1, proof of Theorem 2.3, first paragraph] this theorem is implied by a closedness property of the set of one parameter groups having a ground state.

The purpose of this note is to remark that from the algebraic spectrum condition [2] this closedness property follows naturally for any locally compact abelian group replacing the line.

Let G be a locally compact abelian group and $\mathfrak{A}$ a C^*-algebra with identity I; let $\mathscr{A}$ be the set of all continuous homomorphisms of G into the group of $*$-automorphisms of $\mathfrak{A}$, equipped with the strong topology.

For $\alpha \in \mathscr{A}$, by a representation of $\{\mathfrak{A}, \alpha\}$ we mean a covariant representation (π, U): π is a representation of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$ and U a strongly continuous unitary representation of G on $\mathscr{H}$ s.t. $U(g)\pi(\cdot)U(g)^{-1} = \pi \circ \alpha_g$, $g \in G$.

If ω is an α-invariant state on $\mathfrak{A}$, (π_ω, U_ω) and ξ_ω denote respectively the G.N.S. covariant representation and the associated cyclic vector s.t. $\omega = (\xi_\omega, \pi_\omega(\cdot)\xi_\omega)$ and $U_\omega(g)\pi_\omega(A)\xi_\omega = \pi_\omega(\alpha_g(A))\xi_\omega$; $g \in G$, $A \in \mathfrak{A}$.

Let $\hat{G}$ denote the dual group of G and $K \subset \hat{G}$ a closed set including the identity of $\hat{G}$.

Let $\mathfrak{J}(\alpha, K)$ denote the smallest left ideal in $\mathfrak{A}$ including the set:

$$\mathfrak{B}(\alpha, K) = \{\alpha_f(A)/A \in \mathfrak{A}\,; \quad f \in L^1(G), \quad \hat{f}|K = 0\},$$

where $\alpha_f(A) = \int f(g)\alpha_g(A)d\mu(g)$ and μ is a Haar measure on G.

The following conditions on $\alpha \in \mathscr{A}$ are equivalent:

(i) there exists an α-invariant state ω on $\mathfrak{A}$ with spectrum $U_\omega \subset K$;

(ii) $\mathfrak{J}(\alpha, K) \neq \mathfrak{A}$.