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Free States and Automorphisms of the Clifford Algebra

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Abstract. We study automorphisms of the Clifford algebra which map the set of quasi-free states onto itself. We show that they are quasi-free if the one-particle space is infinite dimensional, and give counter examples in finite dimensions.

In a recent paper [1], Hugenholtz and Kadison have shown an automorphism of the *CAR* or Clifford algebra which maps the set of gauge invariant quasi-free states onto itself to be quasi-free. The same result is known for automorphisms which preserve the set of all quasi-free states [2]. We give simple, alterative proofs of these two results when the one particle space is infinite dimensional and counter-examples when it is not. Because of its economical description of non-gauge invariant free states and of non-unitary Bogoliubov transformations, we have worked in the real Hilbert space formalism of [3]. The connection between this and the complex Hilbert space formalism used in [1] is found in Section 2 of [3].

Let $(H, (,))$ be a real Hilbert space of even or infinite dimension. The C^*-Clifford algebra $\mathfrak{A}(H)$ over H is generated by the range of a linear map $f \to B(f)$ of H into self-adjoint part of $\mathfrak{A}(H)$, satisfying

$$B(f)B(g)+B(g)B(f)=2(f,g)\,. \tag{1}$$

If H' is a subspace of even or infinite dimension we denote by $\mathfrak{A}(H')$ the C^*-subalgebra of $\mathfrak{A}(H)$ generated by $\{B(f)|f \in H'\}$. Every orthogonal transformation $\mathcal{O}$ on H defines a *-automorphism $\alpha_{\mathcal{O}}$ of $\mathfrak{A}(H)$ such that

$$\alpha_{\mathcal{O}}B(f)=B(\mathcal{O}f)\,.$$

Such an automorphism is called quasi-free.

Every anti-hermitian operator A in the unit ball of $B(H)$ defines [3] a state ω_A such that

$$\omega_A(B(f_1)\dots B(f_N))=\begin{cases}0 \text{ if } N \text{ is odd}\\ \sum_{i=2}^{N}(-1)^i\omega_A(B(f_1)B(f_i))\omega_A(B(f_2)\dots\widehat{B(f_i)}\dots B(f_N)) & \text{otherwise}\end{cases} \tag{2}$$

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