Commun. math. Phys. 44, 9—38 (1975)

Killing Tensors and the Separation of the Hamilton-Jacobi Equation*

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Received May 25, 1974

Abstract. This paper investigates the relationship between Killing Tensors and separable systems for the geodesic Hamilton-Jacobi equation in Riemannian and Lorentzian manifolds: locally, a separable system consists of the vector and covector associated with a separable coordinate. It is shown that there are only two types of separable system, those associated with local symmetry groups and those which can be obtained by a simple transformation from orthogonal systems. Some sufficient conditions for existence are given and some global problems are enumerated. The results are illustrated with a demonstration that the existence of separable systems in a certain class of $\{2, 2\}$ space-times is a consequence of the algebraic properties of the Weyl tensor.

§ 1. Introduction

During the century or so following the work of Jacobi, much research in Hamiltonian mechanics was directed towards understanding the separability conditions for the Hamilton-Jacobi equation; this is not surprising since the Hamilton-Jacobi method was (and still is) one of the few analytical tools available. But, with the shift in emphasis in classical mechanics from the local and quantitative to the global and qualitative, interest in this line of research waned, only recently to be revived, first, in general relativity, as a result of Carter's observation that the Hamilton-Jacobi equation for the geodesics in the Kerr solution can be solved by separating the variables [5] and of the subsequent exploitation of this fact in astrophysical calculations (for example, by Bardeen [2]) and, secondly, in quantum mechanics, through the investigation of its relationship to degeneracy and dynamical symmetry in the context of intrinsic quantization (for example, Onofri and Pauri [21]).

The classical research, initiated by Liouville [18] and Stäckel [25] and culminating in a paper by Iarov-Iarovoi [14] in 1964, was aimed, in part, at finding the most general coordinate form of a metric and potential with a separable Hamilton-Jacobi equation. Unfortunately, the use of local "arithmetical" methods did not lead to any real geometrical insights. For example, one feature of the simpler classical and relativistic systems, which is still not properly understood, is that the constants of the motion which arise from the local separation of the Hamilton-Jacobi equation are well behaved in the large. It is likely that this is related to other separability properties of these systems (for instance, to the separability of the associated Schrödinger equations) since the separation of a

* Research supported by the Science Research Council.