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Existence of Phase Transitions for Long-Range Interactions*

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Abstract. Using a theorem about tangent functionals to convex functions, we obtain existence results for phase transitions. In a "large" Banach space of interactions very pathological behavior is found. In spaces of more "reasonable" interactions we obtain co-existing phases which differ in the expectation of a given observable, as well as broken translation invariance due to long-range order. As an example we consider the isotropic Heisenberg model.

1. Introduction

A standard type of problem in statistical mechanics is to describe the translation-invariant equilibrium states for a given interaction. Here we consider the reverse situation: given an invariant state ϱ, we look for an interaction having an invariant equilibrium state $\tilde{\varrho}$ which bears some resemblance to ϱ. In particular, if ϱ exhibits some type of long-range order, we would like $\tilde{\varrho}$ to share this property. This approach leads to some very general existence results for phase transitions, showing that a given type of phase transition occurs for some member of a certain class of interactions.

Our basic point of view will be the identification of invariant equilibrium states with tangent functionals to the pressure on a Banach space of interactions. This identification can be made for classical and quantum lattice and hard-core continuous systems [6–8], but for the sake of simplicity we will mainly consider quantum lattice systems. The main tool, presented in Section 2, is a theorem on approximation of functionals by tangent functionals to a convex function. In Section 3 we consider a "large" Banach space of interactions, finding very pathological behavior. In Section 4 we deal with more "reasonable" interactions, such as pair interactions, finding phase transitions involving the expectation of a given observable, as well as broken translation invariance due to long-range order. Section 5 applies these methods to an important example, the isotropic Heisenberg model, and Section 6 briefly sketches applications to classical systems.

2. Approximation by Tangent Functionals

Let P be a continuous convex function on a real Banach space $\mathscr{X}$. A linear functional $\alpha \in \mathscr{X}^*$ will be called *P-bounded* if for some constant C, $P(\Psi) \geqq \alpha(\Psi) + C$ for all $\Psi \in \mathscr{X}$. We say that α is *tangent to P* at $\Phi \in \mathscr{X}$ if $P(\Phi + \Psi) \geqq P(\Phi) + \alpha(\Psi)$ for

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