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Eigenfunction Expansions for Schrödinger and Dirac Operators with Singular Potentials

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Abstract. The spectral theory (eigenfunction expansion theorem) is developed for Schrödinger and Dirac operators with classes of potentials including $g\exp(-\mu|\boldsymbol{x}|)/|\boldsymbol{x}|^2$ in the Schrödinger case and $g\exp(-\mu|\boldsymbol{x}|)/|\boldsymbol{x}|$ in the Dirac case.

1. Introduction

In the last decade an extensive literature has been devoted to the spectral and scattering theory for Schrödinger and Dirac operators [1–16] and references therein. One of the most powerful methods in this area is that initiated by Ikebe [2]: the generalized eigenfunction expansion method. Although, since the appearence of Ikebe's paper, his proof was generalized and simplified [3–7] it is still rather technical and not general enough to handle singularities like $1/|\boldsymbol{x}|^2$ in the Schrödinger case or $1/|\boldsymbol{x}|$ in the Dirac case [actually in the Dirac case the Yukava potential, $g\exp(-\mu|\boldsymbol{x}|)/|\boldsymbol{x}|$, is the most interesting from the physical point of view].

The aim of this paper is a further generalization and simplification of the proof of the generalized eigenfunction expansion theorem. As a result, our proof applies both to Schrödinger and Dirac cases with singular potentials. Let us point out that we shall not deal here with the most general case we can handle with our method. The reason is that the proof would loose in simplicity requiring minor but long technical details, while no significant gain of relevance to physics is obtained by relaxing, for example, the exponential fall-off at infinity of the potential up to a $|\boldsymbol{x}|^{-\lambda}$ fall-off as far $\lambda > 1$, or allowing space-dimensions greater than three.

Recently, results on the absence of the singular continuous spectrum and the asymptotic completeness for Schrödinger operators with singular potentials have been obtained using other methods by Lavine [9], Pearson and Would [10, 11] and Babbitt and Balslev [12]. Actually though Lavine states the results only for locally L^p potentials with $p > n/2$ (which rules out $1/|\boldsymbol{x}|^2$ singularities) it appears that his method requires, besides conditions at infinity, only $|V|^{1/2}$ to be $H_0^{1/2}$ bounded with relative bound less than one. As far as the Dirac case is concerned, we do not know about results of this type for singular potentials. However, a complex and powerful theory has been created by Kuroda, Agmon, Schechter [13–15] which could possible be used also in the Dirac case[1].

[1] I am grateful to the referee for an extensive comment concerning these points. In particular he pointed out that asymptotic completeness for Schrödinger operators in three dimensions, when $V \in L^1$ and $|V|^{1/2}$ is $H_0^{1/2}$ bounded with relative bound less than one follows [4, Lemma IV. 10] from the fact that $(H+E)^{-1} - (H_0+E)^{-1}$, $E \to \infty$ is trace-class, by an easy extension of the argument in the proof of Theorem II.37 in [4] (this fails, however in the Dirac case or for greater space dimensions).