Commun. math. Phys. 42, 209—220 (1975)

Representations and Inequalities for Ising Model Ursell Functions

Garrett S. Sylvester*

Department of Mathematics, MIT, Cambridge, Mass., Department of Physics, Harvard University, Cambridge, Mass., USA

Received January 13, 1975

Abstract. We describe and investigate representations for the Ursell function u_n of a family of n random variables $\{\sigma_i\}$. The representations involve independent but identically distributed copies of the family. We apply one of these representations in the case that the random variables are spins of a finite ferromagnetic Ising model with quadratic Hamiltonian to show that $(-1)^{\frac{n}{2}+1}u_n(\sigma_1, \ldots, \sigma_n) \geqq 0$ for $n = 2, 4$, and 6 by proving the stronger statement $(-1)^{\frac{n}{2}+1} \left.\frac{\partial^m}{\partial J_{i_1 j_1} \cdots \partial J_{i_m j_m}} Z^{\frac{n}{2}} u_n\right|_{J=0} \geqq 0$ for $n = 2, 4$, and 6, the J_{ij} being coupling constants in the Hamiltonian and Z the partition function. For general n we combine this result with various reductions to show that sufficiently simple derivatives of $(-1)^{\frac{n}{2}+1}Z^{\frac{n}{2}}u_n$, evaluated at zero coupling, are nonnegative. In particular, we conclude that $(-1)^{\frac{n}{2}+1}u_n \geqq 0$ if all couplings are nonzero and the inverse temperature β is sufficiently small or sufficiently large, though this result is not uniform in the order n or the system size. In an appendix we give a simple proof of recent inequalities which bound n-spin expectations by sums of products of simpler expectations.

1. Introduction

The Ursell function $u_n(\sigma_1, \ldots, \sigma_n)$ of a family $\{\sigma_i\}$ of n arbitrary random variables may be defined by means of a generating function as

$$u_n(\sigma_1, \ldots, \sigma_n) = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \ln \mathscr{E}\left(\exp\left[\sum_{i=1}^{n} \lambda_i \sigma_i\right]\right)\Bigg|_{\lambda=0}. \tag{1.1}$$

Here $\mathscr{E}$ is the expectation integral; we assume all the necessary expectations are finite. The Ursell function may be defined recursively by

$$\mathscr{E}(\sigma_1 \sigma_2 \cdots \sigma_n) = \sum_{\mathscr{P}} \prod_{P \in \mathscr{P}} u_{|P|}(\sigma_{p_a}, \sigma_{p_b}, \ldots). \tag{1.2}$$

Here $\mathscr{P}$ is a partition of $\{1, \ldots, n\}$, a set $P \in \mathscr{P}$ has elements p_a, p_b, etc., and $|P|$ denotes the cardinality of P. Finally, $u_n(\sigma_1, \ldots, \sigma_n)$ may be defined explicitly by

$$u_n(\sigma_1, \ldots, \sigma_n) = \sum_{\mathscr{P}} (-1)^{|\mathscr{P}|-1}(|\mathscr{P}|-1)! \prod_{P \in \mathscr{P}} \mathscr{E}\left(\prod_{p \in P} \sigma_p\right), \tag{1.3}$$

where again $\mathscr{P}$ is a partition of $\{1, \ldots, n\}$.

---

* MIT Allen Fellow in Mathematics.

Supported in part by the National Science Foundation under Grant MPS 73-05037.